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Exact asymptotics for the laser linewidth

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Abstract. The Scully–Lamb solution for the diagonal density matrix elements ρ_{nn} of the radiation field of the laser in the photon number basis is used to obtain an expression for the laser linewidth λ, assuming that the Glauber photon correlation function $G^{(2)}(\tau)$ is dominated by a single relaxation rate. In the steady state it is shown that this laser linewidth can be evaluated exactly in terms of the confluent hypergeometric functions ${}_1F_1$ and ${}_2F_2$. Well above threshold we perform a Borel-type summation of the asymptotic series for the hypergeometric functions to establish a remarkably simple representation for the laser linewidth.

1. Introduction

There has been renewed interest in an approximate formula for the linewidth λ of the intensity fluctuation spectrum of a radiation source (Jakeman and Loudon 1991). This formula may be found from the expression for the second-order photon correlation function of the radiation field (Glauber 1963)

$$G^{(2)}(\tau) = \langle a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\rangle \tag{1.1}$$

where $a^\dagger(\tau)$ and $a(\tau)$ are the creation and annihilation operators, respectively, for the laser mode, by assuming that this function is dominated by a single relaxation time. Under these circumstances we can write

$$G^{(2)}(\tau) \approx \langle n\rangle^2 + \left[\langle n(n-1)\rangle - \langle n\rangle^2\right] e^{-\lambda\tau} \tag{1.2}$$

where n is the photon number. Consequently the intensity fluctuation spectral width λ can be obtained from the relation (Jakeman and Pike 1971)

$$\lambda = -\left[\langle n(n-1)\rangle - \langle n\rangle^2\right]^{-1} \frac{\mathrm{d}}{\mathrm{d}\tau} G^{(2)}(\tau)\bigg|_{\tau=0}. \tag{1.3}$$

This assumption is sufficiently accurate for most practical purposes (Pike 1969). In terms of the density matrix $\rho(\tau)$ we also have

$$\begin{aligned} G^{(2)}(\tau) &= \mathrm{tr}_{S,B}\left[\rho(0)a^\dagger(0)a^\dagger(\tau)a(\tau)a(0)\right] \\ &= \mathrm{tr}_{S,B}\left[\rho(0)a^\dagger(0)\exp(\mathrm{i}H\tau/\hbar)a^\dagger(0)a(0)\exp(-\mathrm{i}H\tau/\hbar)a(0)\right] \\ &= \mathrm{tr}_{S,B}\left[\exp(-\mathrm{i}H\tau/\hbar)a(0)\rho(0)a^\dagger(0)\exp(\mathrm{i}H\tau/\hbar)a^\dagger(0)a(0)\right] \end{aligned} \tag{1.4}$$

where S and B denote the laser mode and bath degrees of freedom, respectively.

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We now define

$$\chi_{aa^\dagger}(\tau) \equiv \exp(-\mathrm{i}H\tau/\hbar)a(0)\rho(0)a^\dagger(0)\exp(\mathrm{i}H\tau/\hbar). \tag{1.5}$$

This operator has the same form of time evolution as $\rho(\tau)$ since it is governed by the same Hamiltonian H. Hence if the evolution of the density matrix for the laser mode is described by the well known Scully–Lamb master equation (Scully and Lamb 1967) then we can express the rate of change of

$$\bar{\chi}_{aa^\dagger}(\tau) = \mathrm{tr}_B[\chi_{aa^\dagger}(\tau)] \tag{1.6}$$

in the form

$$\frac{\mathrm{d}}{\mathrm{d}\tau}[\bar{\chi}_{aa^\dagger}(\tau)]_{nn} = -\left[\frac{(n+1)A}{1+(n+1)(B/A)}\right][\bar{\chi}_{aa^\dagger}(\tau)]_{nn} + \left[\frac{nA}{1+n(B/A)}\right][\bar{\chi}_{aa^\dagger}(\tau)]_{n-1,n-1}$$
$$- Cn[\bar{\chi}_{aa^\dagger}(\tau)]_{nn} + C(n+1)[\bar{\chi}_{aa^\dagger}(\tau)]_{n+1,n+1} \tag{1.7}$$

where

$$[\bar{\chi}_{aa^\dagger}(\tau)]_{nn} = \langle n|\bar{\chi}_{aa^\dagger}(\tau)|n\rangle \tag{1.8}$$

$|n\rangle$ is a number state of the laser mode containing n quanta, A and B are the linear gain and self-saturation coefficients, respectively, and C represents cavity damping. Since

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau) = \sum_n \langle n|\frac{\mathrm{d}}{\mathrm{d}\tau}[\chi_{aa^\dagger}(\tau)]a^\dagger(0)a(0)|n\rangle$$
$$= \sum_n n\frac{\mathrm{d}}{\mathrm{d}\tau}[\chi_{aa^\dagger}(\tau)]_{nn} \tag{1.9}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau)\bigg|_{\tau=0} = \sum_n n\bigg\{-\left[\frac{(n+1)A}{1+(n+1)(B/A)}\right]\langle n|a(0)\rho(0)a^\dagger(0)|n\rangle$$
$$+\left[\frac{nA}{1+n(B/A)}\right]\langle n-1|a(0)\rho(0)a^\dagger(0)|n-1\rangle$$
$$+ C(n+1)\langle n+1|a(0)\rho(0)a^\dagger(0)|n+1\rangle - Cn\langle n|a(0)\rho(0)a^\dagger(0)|n\rangle\bigg\}. \tag{1.10}$$

On evaluating the matrix elements we find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau)\bigg|_{\tau=0} = \sum_n n\bigg\{-\left[\frac{(n+1)A}{1+(n+1)(B/A)}\right](n+1)\rho_{n+1,n+1}(0)$$
$$+\left[\frac{nA}{1+n(B/A)}\right]n\rho_{nn}(0) - Cn(n+1)\rho_{n+1,n+1}(0)$$
$$+ C(n+1)(n+2)\rho_{n+2,n+2}(0)\bigg\}. \tag{1.11}$$

If we collect together terms we finally obtain

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau)\bigg|_{\tau=0} = \sum_n n\rho_{nn}(0)\left[\frac{nA}{1+n(B/A)} - (n-1)C\right]. \tag{1.12}$$

Hence, apart from the use of the single-rate relaxation approximation our treatment is exact.

For the Scully–Lamb theory in the steady state we have

$$\rho_{nn}(0) = \frac{u^n}{(1+v)_n\,{}_1F_1(1;1+v;u)} \tag{1.13}$$

where $u = A^2/(BC)$, $v = A/B$ and $(a)_n = \Gamma(n+a)/\Gamma(a)$ denotes the Pochhammer symbol. Jakeman and Loudon (1991) made the approximation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau)\bigg|_{\tau=0} \approx -Cv\sum_{n=0}^{\infty}\left(\frac{n-1}{n-1+v}\right)\rho_{nn}(0) - C\rho_{00}(0) \tag{1.14}$$

which is valid for $v \gg 1$. Further approximations were needed in their analysis. For example, well above threshold, they assumed that

$$\sum_{n=0}^{\infty}\left(\frac{n-1}{v+n-1}\right)\rho_{nn}(0) \approx \frac{\langle n\rangle}{v+\langle n\rangle}. \tag{1.15}$$

Given the narrowness of the distribution this may seem a reasonable approximation but there is no control over the errors introduced. We shall show, however, that it is possible to analyse the behaviour of λ without using such approximations.

2. Exact formula for λ

From the results (1.12) and (1.13) we readily find that

$$\frac{\mathrm{d}}{\mathrm{d}\tau}G^{(2)}(\tau)\bigg|_{\tau=0} = C\left[\frac{u^2}{(1+v)^2\,{}_1F_1(1;1+v;u)}S(v,u) - \langle n(n-1)\rangle\right] \tag{2.1}$$

where

$$S(v,u) \equiv (1+v)^2\sum_{n=1}^{\infty}\frac{n^2u^{n-1}}{(n+v)(1+v)_n}. \tag{2.2}$$

We can express $S(v,u)$ in the alternative form

$$S(v,u) = \sum_{n=0}^{\infty}\frac{(n+1)(2)_n(1+v)_n}{(2+v)_n(2+v)_n}\frac{u^n}{n!}. \tag{2.3}$$

From this result we obtain

$$S(v,u) = {}_2F_2(2,1+v;2+v,2+v;u) + \frac{2u(1+v)}{(2+v)^2}\,{}_2F_2(3,2+v;3+v,3+v;u) \tag{2.4}$$

where ${}_2F_2$ denotes a generalized confluent hypergeometric function.

Next we determine the average values $\langle n \rangle$ and $\langle n(n-1) \rangle$ by first introducing the probability generating function

$$P(z) = \sum_{n=0}^{\infty} \rho_{nn}(0) z^n. \tag{2.5}$$

If we substitute (2) in this equation it is found that

$$P(z) = \frac{{}_1F_1(1; 1+v; uz)}{{}_1F_1(1; 1+v; u)}. \tag{2.6}$$

The application of the standard formula (Luke 1969, p 117)

$$\frac{\mathrm{d}^m}{\mathrm{d}z^m}\, {}_1F_1(a; c; z) = \frac{(a)_m}{(c)_m}\, {}_1F_1(a+m; c+m; z) \tag{2.7}$$

to (2.6) yields the following expression for the factorial moments of the photon-counting distribution:

$$\langle n(n-1)\ldots(n-m+1) \rangle = \frac{m!u^m}{(1+v)_m} \frac{{}_1F_1(1+m; 1+v+m; u)}{{}_1F_1(1; 1+v; u)}. \tag{2.8}$$

Hence we have

$$\langle n \rangle = \frac{u}{(1+v)} \frac{{}_1F_1(2; 2+v; u)}{{}_1F_1(1; 1+v; u)} \tag{2.9}$$

$$\langle n(n-1) \rangle = \frac{2u^2}{(1+v)(2+v)} \frac{{}_1F_1(3; 3+v; u)}{{}_1F_1(1; 1+v; u)}. \tag{2.10}$$

We can now use equations (1.3), (2.1), (2.4), (2.9) and (2.10) to derive an *exact* expression for the spectral linewidth λ in terms of confluent hypergeometric functions.

It is interesting to note that the ${}_1F_1$ functions in (2.9) and (2.10) can all be expressed in terms of the incomplete gamma function $\gamma(a, u)$. To derive these results we first apply the Kummer transformation (Abramowitz and Stegun 1965, p 505)

$${}_1F_1(a; c; u) = \mathrm{e}^u\, {}_1F_1(c-a; c; -u) \tag{2.11}$$

to the confluent hypergeometric function ${}_1F_1(1; 1+v; u)$ and then use the relation (Abramowitz and Stegun 1965, p 509)

$${}_1F_1(v; 1+v; -u) = vu^{-v}\, \gamma(v, u). \tag{2.12}$$

Hence we obtain the identity

$${}_1F_1(1; 1+v; u) = vu^{-v}\mathrm{e}^u \gamma(v, u). \tag{2.13}$$

The application of formula (2.7) to this result yields the further relations

$${}_1F_1(2; 2+v; u) = (1+v)\left[1 + u^{-v}\mathrm{e}^u\gamma(1+v, u)\left(1 - \frac{v}{u}\right)\right] \tag{2.14}$$

$${}_1F_1(3; 3+v; u) = \frac{1}{2}(2+v)\left[(1-v+u) + u^{-v}\mathrm{e}^u\gamma(2+v, u)\left\{1 - \frac{2v}{u} + \frac{v(1+v)}{u^2}\right\}\right]. \tag{2.15}$$

3. Asymptotic behaviour of λ as $u \to +\infty$

The behaviour of λ well above the threshold can be determined by investigating the asymptotic properties of the various confluent hypergeometric functions as $u \to +\infty$, with v fixed. From the general analysis of Luke (1969, pp 201–2) we find that

$$ {}_2F_2(2, 1+v; 2+v, 2+v; u) \sim (1+v)\Gamma(2+v)u^{-1-v}\mathrm{e}^u \sum_{k=0}^{\infty} g_k u^{-k} \tag{3.1}$$

as $u \to +\infty$ with v fixed, where the coefficient g_k satisfies the three-term recurrence relation

$$ (k+1)g_{k+1} - (2k^2 + kv - v)g_k + (k-1)^2(k+v)g_{k-1} = 0 \tag{3.2}$$

with the initial conditions $g_0 = 1$ and $g_{-1} \equiv 0$. From relation (3.2) we obtain the surprisingly simple formula

$$ g_k = -v(k-1)! \qquad (k \geqslant 1). \tag{3.3}$$

This result is consistent with the prototype form for the *late* terms in a general asymptotic series which was proposed by Dingle (1973). The substitution of (3.3) in (3.1) gives the Mth-order asymptotic representation

$$ {}_2F_2(2, 1+v; 2+v, 2+v; u) \sim (1+v)\Gamma(2+v)u^{-1-v}\mathrm{e}^u\left(1 - \frac{v}{u}\sum_{n=0}^{M}\frac{n!}{u^n}\right) \tag{3.4}$$

as $u \to +\infty$, with $M = 0, 1, 2, \ldots$ fixed. In a similar manner it can also be shown that

$$ {}_2F_2(3, 2+v; 3+v, 3+v; u) \sim \frac{1}{2}(2+v)\Gamma(3+v)u^{-1-v}\mathrm{e}^u\left[1 - \frac{(2v+1)}{u} + \frac{v(1+v)}{u^2}\sum_{n=0}^{M}\frac{n!}{u^n}\right] \tag{3.5}$$

as $u \to +\infty$, with $M = 0, 1, 2, \ldots$ fixed.

In order to obtain a Borel-type integral representation (Dingle 1973) for the factorial series in (3.4) and (3.5) we introduce the entire function (Wong 1989)

$$ \Omega(u) = \int_0^u \frac{\mathrm{e}^t - 1}{t}\,\mathrm{d}t \tag{3.6}$$

where $u > 0$. It is possible to express this function in the following alternative forms:

$$ \Omega(u) = u\,{}_2F_2(1, 1; 2, 2; u) \tag{3.7}$$

$$ \Omega(u) = -\gamma - \ln(u) + \mathrm{Ei}(u) \tag{3.8}$$

where γ is the Euler constant and $\mathrm{Ei}(u)$ is the exponential integral (Abramowitz and Stegun 1965, p 228). For large positive values of u the function $\Omega(u)$ has an Mth-order asymptotic representation (Wong 1989)

$$ \Omega(u) \sim \frac{\mathrm{e}^u}{u}\sum_{n=0}^{M}\frac{n!}{u^n} \tag{3.9}$$

as $u \to +\infty$, with $M = 0, 1, 2, \ldots$ fixed.

The application of (3.9) to (3.4) and (3.5) gives the further *closed-form* asymptotic representations

$$_2F_2(2, 1+v; 2+v, 2+v; u) \sim (1+v)\Gamma(2+v)u^{-1-v}e^u\left[1 - ve^{-u}\Omega(u)\right] \tag{3.10}$$

$$_2F_2(3, 2+v; 3+v, 3+v; u) \sim \frac{1}{2}(2+v)\Gamma(3+v)u^{-1-v}e^u \times\left[1 - \frac{(2v+1)}{u} + \frac{v(1+v)}{u}e^{-u}\Omega(u)\right] \tag{3.11}$$

as $u \to +\infty$, with v fixed. These simple formulae appear to be new results for *non-integral* values of v. When $v = 0, 1, 2, \ldots$ both the $_2F_2$ functions in (3.10) and (3.11) can be expressed *exactly* in terms of the function $\Omega(u)$. For example, for the case $v = 1$ we find (Wolfram 1991)

$$_2F_2(2, 2; 3, 3; u) = \frac{4}{u^2}e^u\left[1 - e^{-u}\Omega(u)\right] - \frac{4}{u^2} \tag{3.12}$$

$$_2F_2(3, 3; 4, 4; u) = \frac{9}{u^2}e^u\left[1 - \frac{3}{u} + \frac{2}{u}e^{-u}\Omega(u)\right] + \frac{27}{u^3}. \tag{3.13}$$

We see, therefore, that for $v = 0, 1, 2, \ldots$ we can write down *exact* formulae for the *error* terms in the asymptotic representations (3.10) and (3.11). This feature provides one with some justification for introducing the function $\Omega(u)$ into the analysis. Moreover in this case it could be interesting to construct hyperasymptotics (Berry and Howls 1990) and check the error against the known exact answer.

The asymptotic behaviour of the $_1F_1$ functions in (2.13)–(2.15) can be readily determined by applying the standard expansion (Abramowitz and Stegun 1965, p 263)

$$\gamma(a, u) \sim \Gamma(a) - u^{a-1}e^{-u}\sum_{n=0}^{M}(-1)^n(1-a)_n u^{-n} \tag{3.14}$$

as $u \to +\infty$, with $M = 0, 1, 2, \ldots$ fixed. This procedure yields the representations

$$_1F_1(1; 1+v; u) \sim \Gamma(1+v)u^{-v}e^u \tag{3.15}$$

$$_1F_1(2; 2+v; u) \sim \Gamma(2+v)u^{-v}e^u\left(1 - \frac{v}{u}\right) \tag{3.16}$$

$$_1F_1(3; 3+v; u) \sim \frac{1}{2}\Gamma(3+v)u^{-v}e^u\left[1 - \frac{2v}{u} + \frac{v(1+v)}{u^2}\right] \tag{3.17}$$

as $u \to +\infty$, with v fixed. If these results are substituted in (2.9) and (2.10) we find that

$$\langle n\rangle \sim u\left(1 - \frac{v}{u}\right) \tag{3.18}$$

$$\langle n(n-1)\rangle \sim u^2\left[1 - \frac{2v}{u} + \frac{v(v+1)}{u^2}\right] \tag{3.19}$$

as $u \to +\infty$, with v fixed. Hence we obtain the asymptotic representation

$$\left[\langle n(n-1)\rangle - \langle n\rangle^2\right] \sim v \tag{3.20}$$

as $u \to +\infty$, with v fixed. The leading-order correction term to the formula (3.20) is $-u^{1+v}e^{-u}/\Gamma(v)$.

We now substitute (2.4) in (2.1) and apply the asymptotic relations (3.10), (3.11), (3.15) and (3.19). In this manner we find

$$\frac{d}{d\tau}G^{(2)}(\tau)\bigg|_{\tau=0} \sim Cv[-1 + v\{ue^{-u}\Omega(u) - 1\}] \tag{3.21}$$

as $u \to +\infty$. From this result and equations (1.3) and (3.20) we readily see that the asymptotic behaviour of the spectral linewidth is

$$\lambda \sim C[1 - v\{ue^{-u}\Omega(u) - 1\}] \tag{3.22}$$

as $u \to +\infty$, with v fixed. The application of (3.9) to (3.22) gives the alternative Mth-order asymptotic representation

$$\lambda \sim C\left(1 - v\sum_{n=1}^{M}\frac{n!}{u^n}\right) \tag{3.23}$$

as $u \to +\infty$, with $M = 0, 1, 2, \ldots$ fixed. The first-order representation of (3.23) is

$$\lambda \sim C\left(1 - \frac{v}{u}\right) \tag{3.24}$$

as $u \to +\infty$, with v fixed. This result is consistent with the work of Jakeman and Loudon (1991).

Finally, we check the analysis by evaluating the asymptotic formula (3.22) for the case $v = \frac{1}{2}$ and $u = 20$. It is found that

$$\lambda/C \approx 0.972\,022\,12. \tag{3.25}$$

This result is in excellent agreement with the exact value

$$\lambda/C = 0.972\,022\,233\ldots. \tag{3.26}$$

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